SINGULARITIES OF BLASCHKE NORMAL MAPS OF CONVEX SURFACES

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ABSTRACT. We prove that the difference between the numbers of positive swallowtails and negative swallowtails of the Blaschke normal map for a given convex surface in affine space is equal to the Euler number of the subset where the affine shape operator has negative determinant.

1. INTRODUCTION.

Throughout this Note, we assume that M^2 is a compact oriented 2-manifold without boundary. Let φ be a bundle homomorphism of the tangent bundle TM^2 into a vector bundle E of rank 2 over M^2. A point p on M^2 is called a *singular point* if the linear map $\varphi_p : T_pM \to E_p$ is not bijective. We denote by Σ_φ the set of singular points of φ. We assume that E is orientable, that is, there is a non-vanishing section $\mu : M^2 \to E^* \wedge E^*$, where E^* is the dual vector bundle of E. We now fix a metric $\langle\ ,\ \rangle$ on E. Multiplying a suitable C^∞-function on M^2, we may assume that $\mu(e_1, e_2) = 1$ holds for any oriented orthonormal frame e_1, e_2 on E. By using a positively oriented local coordinate system $(U; u, v)$, the *signed area form* $d\hat{A}$, the signed area density function λ, and the (*un-signed*) *area form* dA are defined by

$$d\hat{A} := \varphi^*\mu = \lambda\, du \wedge dv, \qquad dA := |\lambda|\, du \wedge dv.$$

Both $d\hat{A}$ and dA are independent of the choice of (u, v), and are 2-forms globally defined on M^2. When φ has no singular points, these two forms coincide up to sign. We set

$$M^+ := \{p \in M^2 \setminus \Sigma_\varphi\,;\, d\hat{A}_p = dA_p\}, \qquad M^- := \{p \in M^2 \setminus \Sigma_\varphi\,;\, d\hat{A}_p = -dA_p\}.$$

The singular set Σ_φ coincides with $\partial M^+ = \partial M^-$. A singular point $p(\in \Sigma_\varphi)$ on M^2 is called *non-degenerate* if the derivative $d\lambda$ does not vanish at p. In a neighborhood of a non-degenerate singular point, the singular set can be parametrized as a regular curve $\gamma(t)$ on M^2, called the *singular curve*. The tangential direction of γ is called the *singular direction*. The direction of the kernel of φ_p is called the *null direction*, which is one dimensional. There exists a smooth non-vanishing vector field $\eta(t)$ along γ pointing in the null direction, called the *null vector field*.

Definition 1.1. Take a non-degenerate singular point $p \in M^2$ and let $\gamma(t)$ be the singular curve satisfying $\gamma(0) = p$. Then p is called an A_2-*point* if the null direction $\eta(0)$ is transversal to the singular direction $\dot{\gamma}(0) = d\gamma/dt|_{t=0}$. If p is not an A_2-point, but satisfies that $d(\dot{\gamma}(t) \wedge \eta(t))/dt$ does not vanish at $p = \gamma(0)$, it is called an A_3-*point*, where $\wedge$ is the exterior product on TM^2. We fix an A_3-point

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p. If the angle of the region M^- (resp. M^+) at p with respect to the pull-back metric $ds^2 := \varphi^* \langle\ ,\ \rangle$ is zero, then it is called a *positive* (resp. *negative*) A_3-point. (A_3-points are either positive or negative, see [6]).

We now fix a metric connection D of $(E, \langle\ ,\ \rangle)$. Let $\gamma(t)$ be a regular curve on M^2 consisting only of A_2-points. Take a null vector field $\eta(t)$ such that $(\dot\gamma, \eta)$ is a positive frame of TM^2 along γ. Then

$$\kappa_s(t) = \operatorname{sgn}\bigl(d\lambda(\eta(t))\bigr)\frac{\mu\bigl(\varphi(\dot\gamma(t)), D_t\varphi(\dot\gamma(t))\bigr)}{\langle\varphi(\dot\gamma(t)), \varphi(\dot\gamma(t))\rangle^{3/2}} \tag{1}$$

is called the *singular curvature* of γ at t (see [5] and [6]).

For an oriented orthonormal frame field e_1 ,e_2 of E defined on $U \subset M^2$, there is a unique 1-form ω on U such that $D_X e_1 = -\omega(X)e_2$, $D_X e_2 = \omega(X)e_1$. Then $d\omega$ does not depend on the choice of e_1, e_2, and there is a C^∞-function $K_{\varphi,D}$ on $M^2 \setminus \Sigma_\varphi$ such that

$$d\omega = K_{\varphi,D}\, d\hat A. \tag{2}$$

We call $K_{\varphi,D}$ the *Gaussian curvature* of D with respect to φ. Let $\bar D$ be the pull-back of D on $M^2\setminus\Sigma_\varphi$. Let $\sigma(t)$ be a regular curve on $U\setminus\Sigma_\varphi$ with the arclength parameter t with respect to $ds^2 = \varphi^* \langle\ ,\ \rangle$. We take a unit normal vector $n(t)$ such that $(\dot\sigma, n)$ gives a positive frame on TM^2. On the other hand, we take $\hat n(t) \in E$ such that $(\varphi(\dot\sigma), \hat n)$ gives a positive frame on E. We can define two geodesic curvatures;

$$\kappa_g = ds^2(\bar D_t\dot\sigma(t), n(t)), \qquad \hat\kappa_g = \langle D_t\varphi(\dot\sigma(t)), \hat n(t)\rangle.$$

Here, $\hat\kappa_g(t)$ is well-defined even when $\sigma(t)$ passes through the set Σ_φ. Since $\varphi(n) = \operatorname{sgn}(\lambda)\hat n$, it holds that $\kappa_g = \operatorname{sgn}(\lambda)\hat\kappa_g$. We set $(\bar e_1, \bar e_2) = (\varphi^{-1}(e_1), \varphi^{-1}(e_2))$ if $U \subset M^+$ and set $(\bar e_1, \bar e_2) = (\varphi^{-1}(e_2), \varphi^{-1}(e_1))$ if $U \subset M^-$. Then $(\bar e_1, \bar e_2)$ gives an oriented orthonormal frame on TM^2, and there is a C^∞-function $\theta = \theta(t)$ such that $\dot\sigma = \cos\theta\bar e_1 + \sin\theta\bar e_2$ and $n = -\sin\theta\,\bar e_1 + \cos\theta\,\bar e_2$. Then we get

$$\kappa_g dt = d\theta - (\operatorname{sgn}\lambda)\omega. \tag{3}$$

If the connection D satisfies the condition

$$D_X\varphi(Y) - D_Y\varphi(X) - \varphi([X,Y]) = 0 \tag{4}$$

for all vector fields X, Y on M^2, $(E, \langle\ ,\ \rangle, D, \varphi)$ is called a *coherent tangent bundle*. Under the condition (4), $\bar D$ gives the Levi-Civita connection of ds^2 on $M^2 \setminus \Sigma_\varphi$, and $K_{\varphi,D}$ coincides with the usual Gaussian curvature. We consider a contractible triangular domain $\triangle\mathsf{ABC}$ on $M^2 \setminus \Sigma_\varphi$ such that it lies on the left-hand side of the regular arcs AB, BC, CA which meet transversally at A, B, $\mathsf{C} \in M^2$. By applying the Stokes formula, (2) and (3) yield that

$$\angle\mathsf{A} + \angle\mathsf{B} + \angle\mathsf{C} - \pi = \int_{\partial\triangle\mathsf{ABC}} \kappa_g\, d\tau + \int_{\triangle\mathsf{ABC}} K_{\varphi,D}\, dA, \tag{5}$$

where $\angle\mathsf{A}$, $\angle\mathsf{B}$, $\angle\mathsf{C}$ are the interior angles of the domain $\triangle\mathsf{ABC}$. To prove this, we do not need to assume that $\bar D$ is the Levi-Civita connection. However, we must remember that $K_{\varphi,D}$ is not the usual Gaussian curvature. One crucial point in this setting is that

$$\int_{M^2} K_{\varphi,D}\, d\hat A = \frac{1}{2\pi}\int_{M^2} d\omega$$

coincides with the Euler characteristic χ_E of the vector bundle E. In [6] (see also [5]), the authors gave the following two Gauss-Bonnet type formulas

$$(6)\quad \chi_E = \chi(M^+) - \chi(M^-) + S_+ - S_-, \qquad 2\pi\chi(M^2) = \int_{M^2} K_{\varphi,D}\, dA + 2\int_{\Sigma_\varphi} \kappa_s\, d\tau,$$

under the assumption that $(E, \langle\ ,\ \rangle, D, \varphi)$ is a coherent tangent bundle, where $d\tau$ is the arclength element on the singular set and S_+, S_- are the numbers of positive and negative A_3-points, respectively. After the publication of [6], the authors found that the proof in [6] is based only on the formula (5) and the identity $\kappa_g = \operatorname{sgn}(\lambda)\hat{\kappa}_g$. So we can conclude that the two formulas (6) hold without assuming (4). Moreover, we can generalize these two formulas to φ admitting more general singularities; in other words, Theorem B in [6] holds on φ without assuming (4). If $E = TM^2$, then χ_E coincides with $\chi(M^2) = \chi(M^+) + \chi(M^-)$ in our setting. So we get the following

Theorem 1.2. *Let $\varphi : TM^2 \to TM^2$ be a bundle homomorphism whose singular set consists only of A_2 and A_3-points. Then $2\chi(M^-) = S_+ - S_-$ and $\int_{M^-} K_{\varphi,D}\, d\hat{A} = \int_{\Sigma_\varphi} \kappa_s\, d\tau$ hold.*

Let $f : M^2 \to (N^3, g)$ be an immersion into an orientable Riemannian 3-manifold. Then there is a globally defined unit normal vector field ν along f. We define the shape operator $\varphi : TM^2 \ni v \mapsto -D_v\nu \in TM^2$, as a bundle homomorphism, where D is the Levi-Civita connection of (N^3, g). A singular point of φ is called an *inflection point* of f. We get the following

Corollary 1.3 (A generalization of the Bleeker-Wilson formula)**.** *Suppose that the shape operator admits only A_2 and A_3-points. Then $2\chi(M^-) = I_+ - I_-$ holds, where I_+* (*resp. I_-*) *is the number of positive* (*resp. negative*) *A_3-inflection points.*

The original formula was for the case $N^3 = \boldsymbol{R}^3$ (see [2]). In [7], the authors pointed out that the formula holds for space forms. Also, they gave in [7] several applications of (6) under the assumption (4). However, now we can remove (4), and we get also the results that follow here.

2. Rotation of vector fields.

We fix a Riemannian metric ds^2 on M^2. There is a unique 2-form μ on M^2 such that $\mu(e_1, e_2) = 1$ where e_1, e_2 is a local oriented orthonormal frame field on M^2. Let X be a vector field on M^2. The C^∞-function $\operatorname{rot}(X) := \mu(D_{e_1}X, D_{e_2}X)$ defined on M^2 is called *the rotation* of X, where D is the Levi-Civita connection of (M^2, ds^2). Consider a bundle homomorphism $\varphi : TM^2 \ni v \mapsto D_vX \in TM^2$. The singular set Σ_X of φ coincides with the zeros of $\operatorname{rot}(X)$, called the set of *irrotational points*. Moreover, an A_3-singular point is called an *irrotational cusp*. In fact, if $M^2 = \boldsymbol{R}^2$ is the Euclidean plane, then X induces a map $\tilde{X} : \boldsymbol{R}^2 \to \boldsymbol{R}^2$, and A_3 (resp. A_2) points correspond to cusps (resp. folds) of $\tilde{X}$ (see [7]). Suppose that X admits only A_2 and A_3-irrotational points. Then Σ_X consists of a finite disjoint union of closed regular curves $\gamma_1, \dots, \gamma_m$ on M^2 such that M^+ lies in the left hand side of each γ_j. Then the singular curvature on γ_j is given by $\kappa_s := \mu(\dot{X}, \ddot{X})/|\dot{X}|^3$ (we propose to call it the *irrotational curvature*), where $\dot{X} = D_{\dot{\gamma}_j(t)}X$ and $\ddot{X} = D_{\dot{\gamma}_j(t)}\dot{X}$. The following assertion follows directly from Theorem 1.2.

Proposition 2.1. *Suppose that X admits only A_2 and A_3-irrotational points. Then it holds that*

$$2\chi(M^-) = C_+ - C_-, \qquad \int_{M^-} K_{\varphi,D} d\hat{A} = \int_{\Sigma_X} \kappa_s d\tau,$$
$$M^- := \left\{ p \in M^2 \,;\, \mathrm{rot}(X)_p < 0 \right\},$$

where C_+ (resp. C_-) is the number of positive (resp. negative) irrotational cusps.

3. Singularities of Blaschke normal maps on convex surfaces.

Let S^2 be a 2-sphere and $f : S^2 \to \boldsymbol{R}^3$ a strictly convex embedding. In affine differential geometry, it is well-known that there are a transversal vector field ξ along f, a torsion free connection ∇, a bundle homomorphism $\alpha : TS^2 \to TS^2$ (called the *affine shape operator*), and a positive definite symmetric covariant tensor h such that (cf. [4]) $D_X Y = \nabla_X Y + h(X,Y)\xi$ and $D_X \xi = -\alpha(X)$ for any vector fields X, Y on S^2, where D is the canonical affine connection on $\boldsymbol{R}^3$. Moreover, such a structure (ξ, ∇, α, h) is uniquely determined up to a constant multiplication of ξ. Here ξ induces a map $\tilde{\xi} : S^2 \to \boldsymbol{R}^3$ called the *Blaschke normal map*. It is obvious that the singular points of α coincides with those of $\tilde{\xi}$.

Lemma 3.1. *The Blaschke normal map $\tilde{\xi}$ is a wave front (cf. [1] for the definition of wave front).*

Proof. Consider a non-zero section $L : S^2 \ni p \mapsto (\tilde{\xi}_p, \nu_p) \in T^*\boldsymbol{R}^3 = \boldsymbol{R}^3 \times (\boldsymbol{R}^3)^*$, where $\nu : S^2 \to (\boldsymbol{R}^3)^*$ is the map into the dual vector space $(\boldsymbol{R}^3)^*$ of $\boldsymbol{R}^3$ such that $\nu_p(\tilde{\xi}_p) = 1$ and $\nu_p(df(T_pS^2)) = \{0\}$ for each $p \in S^2$. Take a local coordinate system (u_1, u_2) of S^2. Then we have that

$$\begin{aligned} \nu_{u_i}(f_{u_j}) = D_{\partial_i}\nu(f_{u_j}) &= \nu(D_{\partial_i} f_{u_j}) \\ &= -\nu\left(\nabla_{\partial_i}\partial_j + h(\partial_i, \partial_j)\tilde{\xi}\right) = -h(\partial_i, \partial_j) \quad (i, j = 1, 2), \end{aligned}$$

where $\partial_i := \partial/\partial u_i$ and $f_{u_i} := df(\partial_i)$. Since h is positive definite, ν_{u_1}, ν_{u_2} are linearly independent. Moreover, $\nu, \nu_{u_1}, \nu_{u_2}$ are also linearly independent, since $\nu(df(T_pS^2)) = 0$. In particular, L induces a Legendrian immersion of S^2 into the projective cotangent bundle $P(T^*\boldsymbol{R}^3)$ of $T^*\boldsymbol{R}^3$. □

By applying the criteria of cuspidal edges and swallowtails (cf. [7], A_2 and A_3-points correspond to the cuspidal edges and swallowtails of the Blaschke normal map $\tilde{\xi}$. So we get the following

Theorem 3.2. *Suppose that $\tilde{\xi}$ admits only cuspidal edges and swallowtails. Then $2\chi(M^-) = S_+ - S_-$ holds, where $M^- := \{p \in S^2 \,;\, \det(\alpha(p)) < 0\}$ and S_+ (resp. S_-) is the number of positive (resp. negative) swallowtails of $\tilde{\xi}$.*

A different formula for $S_+ + S_-$ is given by Izumiya-Marer [3].

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